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Edge-Directions of Standard Polyhedra with Applications to Network Flows

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Abstract. Recent results show that edge-directions of polyhedra play an important role in (combinatorial) optimization; in particular, a d-dimensional polyhedron with $|D|$ distinct edge-directions has at most $O(|D|^{d-1})$ vertices. Here, we obtain a characterization of the directions of edges that are adjacent to a given vertex of a standard polyhedron of the form $P = \{x : Ax = b, l \leqslant x \leqslant u\}$, tightening a standard necessary condition which asserts that such directions must be minimal support solutions of the homogenous equation $Ax = 0$ which are feasible at the given vertex. We specialize the characterization for polyhedra that correspond to network flows, obtaining a graph characterization of circuits which correspond to edge-directions. Applications to partitioning polyhedra are discussed.

Key words: Edge-directions, network flows, polyhedra

1. Introduction

Dantzig's classical Simplex method [2] has been key in the study and solution of linear optimization problems over polytopes for over half a century. The method is based on moving along edges of the underlying polytope. Still, with few exceptions (e.g. [1] and references therein), the focus of the study of polyhedra in the context of linear optimization has been on the vertices and on the facets of the underlying polytope. The conceptual idea of moving along distinctive directions underlines a more recent study [10], which shows how a linear combinatorial optimization oracle can be generated from an augmentation oracle.

The role of edge-directions of polyhedra in (combinatorial) optimization has been explored more recently in [3, 4, 5, 11]. In particular, [4] and [11] derived (independently) a unification of classic conditions (quasi-convexity and Schur convexity) that suffice for a function over a polytope P to

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obtain a maximum at one of $P's$ vertices. The unified condition requires that the function is *edge-quasi-convex on* P, that is, that it is quasiconex along lines that are parallel to the edges of P. Also, results in [3] were enhanced in [5] to develop an algorithm that enumerates all the vertices of a polytope. The input for the algorithm consists of a list of vectors that contains directions of all edges of the underlying polytope. In particular, one gets a polynomial bound on the number of vertices of a polytope in terms of the number of distinct edge-directions it has and in terms of its dimension. The vertex enumeration algorithm facilitates the efficient solution of convex combinatorial optimization problems. For example, it is shown in [5] and [8], respectively, how the edge-directions of network polyhedra can be used to solve efficiently certain partitioning problems and to determine the Nash solution of partition bargaining games.

In Section 2 we derive the main results of this paper which concerns the derivation of a condition that is necessary and sufficient for edge-directions of standard polyhedra, tightening a standard necessary condition. As a result, we obtain bounds on the number of edge-directions. The results are specialized in Section 3 to network polyhedra.

2. Edge-Directions of Polyhedra in Standard Form

Throughout this section, we consider *the polyhedron in standard form* $P=\{x\in\mathbb{R}^n : Ax=b, l\leqslant x\leqslant u\}$ defined by $A\in\mathbb{R}^{m\times n}$ (*the coefficient-matrix*), $b\in\mathbb{R}^m$ (*right-hand side*) and $l\in(\mathbb{R}\cup\{-\infty\})^m$ and $u\in(\mathbb{R}\cup\{+\infty\})^n$ (*lower and upper bounds*).

A *circuit* (of the matrix A) is a nonzero solution $z\in\mathbb{R}^n$ of the equality system $Az=0$, whose support $\text{supp}(z):=\{j : z_j\neq 0\}$ is inclusion-minimal and whose ℓ_∞ norm $\|z\|_\infty$ equals 1. Note that if z is a circuit then no scalar multiple of it other than $\pm z$ is. Clearly, the number of circuits is at most $2\sum_{k=1}^{m+1}\binom{n}{k}$. It is well known (cf. [7, ex. 10.14, p. 506]) that any nonzero real solution of $Az=0$ has a *conformal circuit decomposition*, i.e. can be expressed as $z=\sum_i \alpha_i z^i$, where each α_i is a positive real and the z^is are distinct circuits, each satisfying $z^i_j z_j>0$ for all $j\in\text{supp}(z^i)$. Of course, in such a decomposition, there are no pairs of circuits which are the negative of each other.

A *direction* of a face F of P is any nonzero scalar multiple of $y-x$ for vectors x and y in F. Consider the equivalence relation $\sim$ on $R^n\backslash\{0\}$, where $d\sim d'$ when d' is a (nonzero) scalar multiple of d. Evidently, the set of directions of a face of P is the union of $\sim$-equivalence classes to which we refer as $\sim$-directions. In particular, a 1-dimensional face has a single $\sim$-direction.

The next lemma shows that every circuit in a conformal decomposition of a direction of a face of P, is itself, a direction of that face.

LEMMA 1. *Suppose x and y are in P and $y-x$ has a conformal circuit decomposition $y-x=\sum_{i=1}^{q}\alpha_i z^i$ where each α_i is a positive real and the z^i's are distinct circuits of A, each satisfying $z^i_j(y-x)_j>0$ for all $j\in supp(z^i)$. Then for every $i, x+\alpha_i z^i$ is in each face of P that contains both x and y.*

Proof. For each t and j:

(i) if $z^t_j>0$, then $(y-x)_j>0$, implying that $z^i_j\geqslant 0$ for each i and $l_j\leqslant x_j\leqslant x_j+\alpha_t z^t_j\leqslant x_j+\sum_i\alpha_i z^i_j=y_j\leqslant u_j$,
(ii) if $z^t_j<0$, then $(y-x)_j<0$, implying that $z^i_j\leqslant 0$ for each i and $l_j\leqslant y_j=x_j+\sum_i\alpha_i z^i_j\leqslant x_j+\alpha_t z^t_j\leqslant x_j\leqslant u_j$, and
(iii) if $z^t_j=0$, then $x_j+\alpha_j z^t_j=x_j$ and $l_j\leqslant x_j=x_j+\alpha_t z^t_j\leqslant u_j$.

So, in either case, $l_j\leqslant(x+\alpha_t z^t)_j\leqslant u_j$. As we trivially have $A(x+\alpha_t z^t)=Ax=b$, we conclude that $x+\alpha_t z^t\in P$.

Next, let F be a face of P that contains both x and y. As

$$\frac{1}{q}\sum_{i=1}^{q}(x+\alpha_i z^i)=\left(1-\frac{1}{q}\right)x+\frac{1}{q}\left(x+\sum_{i=1}^{q}\alpha_i z^i\right)=\left(1-\frac{1}{q}\right)x+\frac{1}{q}y\in F,$$

we have from the extremality of F that $x+\alpha_i z^i\in F$ for each i. □

For $x\in\mathbb{R}^n$ let $\text{float}(x):=\{j: l_j<x_j<u_j\}$. We next record two standard results about in relationships of vertices and edge-directions to circuits and floats, see [7]. They provide, respectively, a characterization of vertices and a necessary condition for edge-directions. The latter is tightened in the forthcoming Theorem 5 to a characterization of edge-directions.

PROPOSITION 2. *A vector x in P is a vertex of P if and only if there is no circuit z with $supp(z)\subseteq float(x)$.*

Proof. If there is a circuit z with $\text{supp}(z)\subseteq\text{float}(x)$ then $x\pm\epsilon z\in P$ for small $\epsilon>0$. Consequently, $x=\frac{1}{2}[(x+\epsilon z)+(x-\epsilon z)]$ is not a vertex. Conversely, suppose x is not a vertex. Then $x=\frac{1}{2}(x^1+x^2)$ for some distinct $x^1,x^2\in P$; of course, $A(x^2-x^1)=0$. Consider a conforming circuit decomposition of x^2-x^1, say $x^2-x^1=\sum_i\alpha_i z^i$. Now, if $z^1_j>0$, then $(x^2-x^1)_j>0$, that is, $x^2_j>x^1_j$; as $x_j=\frac{1}{2}(x^1_j+x^2_j)$ we then have $l_j\leqslant x^1_j<x_j<x^2_j\leqslant u_j$. Similarly, if $z^1_j<0$, then $x^2_j<x^1_j$ and $l_j\leqslant x^2_j<x_j<x^1_j\leqslant u_j$. So, whenever $z^1_j\neq 0$ we have that $j\in\text{float}(x)$, that is, $\text{supp}(z^1)\subseteq\text{float}(x)$. □

PROPOSITION 3. *Every direction of a 1-dimensional face of P is a scalar multiple of a circuit of A.*

Proof. Consider a pair of distinct vectors, say x and y, which are in a 1-dimensional face of P, say E. Then, $A(y-x)=0$ and we may consider a conformal circuit decomposition of $y-x$, say $y-x=\sum_{i=1}^{q}\alpha_i z^i$. By Lemma 1, $x+\alpha_i z^i\in E$ for each i, implying that $z^i=(\alpha_i)^{-1}[(x+\alpha_i z^i)-x]$ is a scalar multiple of the difference between two vectors in E. As $\dim E=1$, we conclude that each pair of z^i's, if any, are linearly dependent. It follows that necessarily $1\leqslant q\leqslant 2$, and if $q=2$, then $z_2=-z_1$. But, the latter cannot happen because $z^i_j(y-x)_j>0$ for each i and j with $z^i_j\neq\emptyset$. So, $q=1$, implying that, $y-x$ is a scalar multiple of a circuit. □

Proposition 3 shows that the $\sim$-directions of a 1-dimensional face of P correspond to circuits. As circuits come in pairs which are the negative of each other and each pair is determined by the set of its nonzero variables, we get the following bound of $\sim$-directions of 1-dimensional faces.

COROLLARY 4. *The number of $\sim$-directions of 1-dimensional faces of P is bounded by $\sum_{k=1}^{m+1}\binom{n}{k}$.*

The necessary condition of Proposition 3 for directions of a 1-dimensional face is next tightened to a condition which is both necessary and sufficient.

THEOREM 5. *Suppose x is a vertex of P. Then $y\in P\backslash\{x\}$ lies in a 1-dimensional face of P that contains x if and only if y has a representation $y=x+\alpha z$ with $\alpha>0$ and with z as a circuit of A for which there is no circuit z' of A satisfying:*

$$\operatorname{supp}(z')\neq\operatorname{supp}(z), \tag{1}$$

$$\operatorname{supp}(z')\subseteq\operatorname{supp}(z)\cup\operatorname{float}(x), \tag{2}$$

$$z'_j z_j\geqslant 0 \quad \text{for each } j\notin\operatorname{float}(x). \tag{3}$$

Proof. Suppose $y\in P\backslash\{x\}$ lies in a 1-dimensional face E of P that contains x. By Proposition 3, for some $\alpha>0$ and circuit z of A, $y-x=\alpha z$. We will assume the existence of a circuit $z'\neq z$ of A which satisfies (1)–(3), and will establish a contradition to the assumption that dim $E=1$.

Let

$$\beta\equiv\min\left\{\frac{\alpha z_j}{z'_j}: j\in[\operatorname{supp}(z')\cap\operatorname{supp}(z)]\backslash\operatorname{float}(x)\right\}$$

and

$$\gamma\equiv\min\left\{\frac{\min\{(u_j-x_j),(x_j-l_j)\}}{|z'_j|}: j\in\operatorname{supp}(z')\cap\operatorname{float}(x)\right\}$$

(with the standard convention where $\min\emptyset = +\infty$). Condition (3) assures that $\beta > 0$. Also, clearly, $\gamma > 0$. So, $\delta \equiv \min\{\beta, \gamma, 1\} > 0$. We will show that the vectors $\frac{1}{2}(x + y \pm \delta z')$ are in P. To see that $l \leqslant \frac{1}{2}(x + y \pm \delta z') \leqslant u$, we recall that $l \leqslant \frac{1}{2}(x + y) \leqslant u$ and consider four cases, which are exhaustive by condition (2):

Case I: $z'_j > 0$ and $j \in \text{supp}(z) \setminus \text{float}(x)$: In this case, $0 < \delta z'_j \leqslant \alpha z_j$; thus,

$$l_j \leqslant \frac{1}{2}(x+y)_j < \frac{1}{2}(x+y+\delta z')_j \leqslant \frac{1}{2}(x+y+\alpha z)_j = y_j \leqslant u_j$$

and

$$l_j \leqslant x_j = \frac{1}{2}(x+y-\alpha z)_j \leqslant \frac{1}{2}(x+y-\delta z')_j < \frac{1}{2}(x+y)_j \leqslant u_j.$$

Case II: $z'_j < 0$ and $j \in \text{supp}(z) \setminus \text{float}(x)$: In this case, $0 > \delta z'_j \geqslant \alpha z_j$; thus,

$$l_j \leqslant y_j = \frac{1}{2}(x+y\alpha z)_j \leqslant \frac{1}{2}(x+y+\delta z')_j < \frac{1}{2}(x+y)_j \leqslant u_j$$

and

$$l_j \leqslant \frac{1}{2}(x+y)_j < \frac{1}{2}(x+y-\delta z')_j \leqslant \frac{1}{2}(x+y-\alpha z)_j = x_j \leqslant u_j.$$

Case III: $z'_j \neq 0$ and $j \in \text{float}(x)$: In this case, $\pm\delta z'_j \leqslant \delta|z'_j| \leqslant \min\{x_j - l_j, u_j - x_j\}$; thus,

$$\begin{aligned} l_j &= \frac{1}{2}x_j + \frac{1}{2}l_j - \frac{1}{2}(x_j - l_j) \leqslant \frac{1}{2}x_j + \frac{1}{2}y_j \pm \frac{1}{2}\delta z'_j \\ &\leqslant \frac{1}{2}x_j + \frac{1}{2}u_j + \frac{1}{2}(u_j - x_j) = u_j. \end{aligned}$$

Case IV: $z'_j = 0$: In this case, $\frac{1}{2}(x + y \pm \delta z')_j = \frac{1}{2}(x+y)_j$ and, $l_j \leqslant \frac{1}{2}(x + y \pm \delta z'_j) \leqslant u_j$.

Trivially $A[\frac{1}{2}(x + y \pm \delta z')] = b$ (as $Ax = Ay = b$ and z' is a circuit of A). Hence, $\frac{1}{2}(x + y \pm \delta z') \in P$. Now, as $\frac{1}{2}(x + y) \in E$ (x and y are in E) and $\frac{1}{2}(x + y) = \frac{1}{2}\left[\frac{1}{2}(x+y+\delta z') + \frac{1}{2}(x+y-\delta z')\right]$, we conclude that $\frac{1}{2}(x + y + \delta z') \in E$ and therefore, $z' = \frac{2}{\delta}\left\{\frac{1}{2}(x+y+\delta z') - \frac{1}{2}(x+y)\right\}$ is proportional to the difference of two vectors in E. As condition (1) assures that z and z' are linearly independent, we get a contradiction to the assumption that dim $E = 1$.

We next establish the sufficiency condition for $y \in P\setminus\{x\}$ to be in a 1-dimensional face of P that contains x. So, assume that $y = x + \alpha z$ is in P, where $\alpha > 0$ and z is a circuit of A such that there is no circuit z' of A satisfying conditions (1)–(3).

Consider the vector c in $\mathbb{R}^n$ with

$$c_j = \begin{cases} 0 & \text{if } x_j \neq y_j, \\ 0 & \text{if } l_j < x_j = y_j < u_j \\ 0 & \text{if } l_j = x_j = y_j = u_j \\ +1 & \text{if } l_j < x_j = y_j = u_j, \text{ and} \\ -1 & \text{if } l_j = x_j = y_j < u_j, \end{cases}$$

and let F be the face of P consisting of the maximizers of $c^T w$ over $w \in P$. For $w \in \mathbb{R}^n$,

$$\begin{aligned} c^T x - c^T w &= (+1)\left[\sum_{l_j < x_j = y_j = u_j} (x_j - w_j)\right] + (-1)\left[\sum_{l_j = x_j = y_j < u_j} (x_j - w_j)\right] \\ &= \sum_{l_j < x_j = y_j = u_j} (u_j - w_j) + \sum_{l_j = x_j = y_j < u_j} (w_j - l_j). \end{aligned} \tag{4}$$

As the terms on the right-hand side of (4) are nonnegative for each w satisfying $l \leqslant w \leqslant u$, we have that $c^T w \leqslant c^T x$ for all $w \in P$, implying that $x \in F$; further, $w \in P$ is in F if and only if

$$(l_j < x_j = y_j = u_j) \Rightarrow (w_j = u_j = x_j) \tag{5}$$

and

$$(l_j = x_j = y_j < u_j) \Rightarrow (w_j = l_j = x_j). \tag{6}$$

In particular, we conclude that $y \in F$. Further, for j with $l_j = u_j$ and for $w \in \mathbb{R}^n$ satisfying $l \leqslant w \leqslant u$, we have that $w_j = l_j = u_j = x_j$. This conclusion and the characterization of vectors in P which are in F by (5)–(6) imply that for all $w \in F$:

$$[(w_j \neq x_j) \quad \text{and} \quad (x_j = y_j)] \Rightarrow [l_j < x_j < u_j] \Leftrightarrow [j \in \text{float}(x)]. \tag{7}$$

We will prove that $\dim(F) = 1$, by contradiction, demonstrating that if $\dim(F) \neq 1$, then there exists a circuit z' of A satisfying conditions (1)–(3). So, assume that $\dim(F) \neq 1$. As $z = \alpha^{-1}(y - x) \in$ tng F, standard arguments (e.g., [6]) show that for some $w \in F \setminus \{x\}$, $w - x$ is not a scalar multiple of z. As $A(w - x) = 0$, $w - x$ has a conformal circuit decomposition, say $w - x = \sum_{i=1}^{q} \alpha_i z^i$. By Lemma 1 (applied to x and w), $x + \alpha_i z^i \in F$ for each i. As

$w-x$ is not a scalar multiple of z, we conclude the existence of $\alpha'>0$ and a circuit z' of A with supp $(z')\neq$ supp (z) such that $x+\alpha' z'\in F$ and

$$(z'_j\neq 0)\Rightarrow (w_j-x_j)z'_j>0; \tag{8}$$

in particular, z' satisfies condition (1). As $w\in F$ satisfies (8) and (7),

$$[(z'_j\neq 0) \quad \text{and} \quad (z_j=0)]\Rightarrow[(w_j\neq x_j) \quad \text{and} \quad (x_j=y_j)]\Rightarrow[j\in \text{float}(x)], \tag{9}$$

establishing that z' satisfies condition (2). Finally, to establish (3), assume that $z'_j\neq 0$ and $j\notin \text{float}(x)$. Now, if $z'_j>0$, (8) and $w\leqslant u$ imply that $u_j\geqslant w_j>x_j$; as $j\notin\text{float}(x)$, necessarily $x_j=l_j$. Hence, $y_j\geqslant l_j=x_j$ and therefore $z_j=\alpha^{-1}(y-x)_j\geqslant 0$. Similarly, if $z'_j<0$, (8) and $w\geqslant l$ imply that $l_j\leqslant w_j<x_j$ and therefore $x_j=u_j$ and $y_j\leqslant u_j=x_j$, assuring that $z_j=\alpha^{-1}(y-x)_j\leqslant 0$. This proves that when $z'_j\neq 0$ and $j\notin\text{float}(x)$, $z'_jz_j\geqslant 0$, establishing condition (3). □

Proposition 2 assures that if x is a vertex of P and z' is a circuit of A, then supp$(z')\nsubseteq$ float(x). Consequently, condition (2) in the statement of Theorem 5 implies that

$$\text{supp}(z')\cap\text{supp}(z)\neq\emptyset. \tag{10}$$

Also, given $x\in P$ and $z\in\mathbb{R}^n$, $x+\alpha z\in P$ for some $\alpha>0$ if and only if $Az=0$ and

$$[(z_j>0)\Rightarrow(x_j<u_j)] \quad \text{and} \quad [(z_j<0)\Rightarrow(x_j>\ell_j)], \tag{11}$$

yielding a simple necessary condition for a circuit z to be a direction of a 1-dimensional face that contains x.

3. Edge-Directions of Network Polyhedra

We next examine edge-directions of network polyhedra. Henceforth, we consider a finite *directed graph* $G=(N,U)$ *without loops* where $N=\{1,\dots,|N|\}$ is the (finite) set of *nodes* and $U\subseteq N\times N\setminus\{(i,i):i\in N\}$ is the set of *arcs*. If $e=(r,s)\in U$, we say that nodes r and s *occur* in e. Henceforth, we assume that the elements of U are enumerated by $1,\dots,|U|$; in particular, vectors in $(\mathbb{R}\cup\{-\infty\}\cup\{+\infty\})^{|N|}$ and $(\mathbb{R}\cup\{-\infty\}\cup\{+\infty\})^{|U|}$ are identified with the corresponding functions on N and U, and coordinates of vectors and matrices of corresponding size are indexed by arcs. For example, the *node-arc incidence-matrix* of G is the $|N|\times|U|$ matrix Γ with $\Gamma_{\text{re}}=-1$ if $e=(r,i)$ for some $i\in N\setminus\{r\}$, $\Gamma_{\text{se}}=1$ if $e=(i,s)$ for some $i\in N\setminus\{s\}$, and $\Gamma_{\text{ie}}=0$ if i does not occur in e. When A is the node-arc incidence-matrix of a graph G, we refer to P as a *network polyhedron* and to elements of P as *network flows*; in particular, we let $|N|=m$ and $|U|=n$. Also, circuits of A are then called *cycles*; the coordinates of a cycle are $-1, 0$ or $+1$ and cycles correspond to (permutation-invariant) sequences of nodes/edges.

Of course, Proposition 3 and Corollary 4 specialize to network polyhedra. Further, the representation of cycles via sequences of nodes yields the following modification of the bound of Corollary 4.

COROLLARY 6. *The number of $\sim$-directions of 1-dimensional faces of (the network polyhedron) P is bounded by* $\frac{1}{2}\sum_{k=2}^{m}\binom{m}{k}(k-1)!$.

Proof. The bound follows from a count of the of distinct sequences of nodes, while factoring out permutation- and direction-invariance of representations of cycles via sequences of nodes. □

We will specialize Theorem 5 to network polyhedra. For this purpose, we shall need the following result.

LEMMA 7. *Given a cycle z and a set of arcs U', the following are equivalent:*

(a) There exists a cycle z' such that:

(i) $supp(z') \subseteq supp(z) \cup U'$,
(ii) $supp(z') \cap supp(z) \neq \emptyset$,
(iii) $z'_e z_e \geqslant 0$ *for each* $e \in U \setminus U'$, *and*
(iv) $supp(z') \neq supp(z)$.

(a-) The conditions of (a) excluding (iii).

(b) There is an enumeration $e_1, \dots, e_q, e_{q+1} = e_1$ of the arcs in $supp(z)$ such that a pair of arcs in this sequence are consecutive if and only if there is a node that occurs in both, and there exist a positive integer $1 \leqslant s < q$ and arcs $h_1, \dots, h_s$ in $U' \setminus \{e_1, \dots, e_q\}$ such that for some $1 \leqslant m < p \leqslant q$,

(i) *no arc in $\{h_2, \dots, h_{s-1}\}$ is adjacent to any arc in $\{e_1, \dots, e_q\}$,*
(ii) *h_1 is adjacent to arcs e_m and e_{m+1}, and*
(iii) *h_s is adjacent to arcs e_p and e_{p+1}.*

(c) There exist cycles z^1 and z^2 with

(i) $z = z^1 + z^2$,
(ii) $supp(z^1) \cap supp(z^2) \subseteq U' \setminus supp(z)$, *and*
(iii) $z^t_e z_e \geqslant 0$ *for* $t = 1, 2$ *and* $e \in U$.

(c-) The conditions of (c) excluding (iii).

Proof. (a) $\Rightarrow$ (a-) and (c) $\Rightarrow$ (c-): These implications are trite.

(a-) $\Rightarrow$ (b): Assume that (a-) holds. As $\mathrm{supp}(z') \cap \mathrm{supp}(z) \neq \emptyset$, there exists an edge g in U with $z'_g \neq 0$ and $z_g \neq 0$. By possibly replacing z' with $-z'$, we conclude the existence of a cycle z' which satisfied conditions (i), (ii), (iv) of (a) and $z'_g = z_g$ for some $g \in U$; in particular, (iv) and the fact that z' and z are cycles imply that $\mathrm{supp}(z') \backslash \mathrm{supp}(z) \neq \emptyset$.

Now, standard results show that the supports of z and z' can be ordered, respectively, as sequences $e_1, \dots, e_q$ and $f_1, \dots, f_{q'}$, such that a pair of arcs belonging to any one of the two sequences are consecutive if and only if there is a node that occurs in both, and where $e_{q+1} \equiv e_1$ and $f_{q'+1} = f_1$. Without loss of generality we may assume that $e_1 = f_1 = g$. As the supports of z' and z do not coincide, there is an integer $2 \leqslant i \leqslant \min\{q, q'\}$ with $f_i \neq e_i$. Let $m + 1 \geqslant 2$ be the first such integer; in particular,

(i) $e_i = f_i$ for $i = 1, \dots, m$, and
(ii) $f_{m+1} \notin \{e_1, \dots, e_q\}$ and f_{m+1} is adjacent to $e_m = f_m$ and to e_{m+1}.

It follows for some integers p and r satisfying $m < p \leqslant q$ and $m < r \leqslant q$:

(iii) $\{f_{m+1}, f_{m+2}, \dots, f_r\} \cap \{e_1, \dots, e_q\} = \emptyset$,
(iv) no arc in $\{f_{m+2}, \dots, f_{r-1}\}$ is adjacent to any arc in $\{e_1, \dots, e_q\}$, and
(v) f_r is adjacent to arcs e_p and e_{p+1}.

See Figure 1 for an example with $m = 2, r = 6$ and $p = 8$. (It is noted that $p = r = m + 1$ is not excluded—in this case f_{m+1} as the inverse of e_{m+1}, that is, the same pair of two nodes occurs in f_{m+1} and in e_{m+1}, but with a different orientation.) For $e \in \{f_{m+1}, f_{m+2}, \dots, f_r\}, z'_e \neq 0$ and $z_e = 0$ and therefore condition (i) of (a) assures that such e is in U'. So, $\{f_{m+1}, \dots, f_r\} \subseteq U' \backslash \{e_1, \dots, e_q\}$. It follows that $s \equiv r - m$ and $h_1 \equiv f_{m+1}, h_2 \equiv f_{m+2}, \dots, h_s \equiv f_r$ satisfy (b).

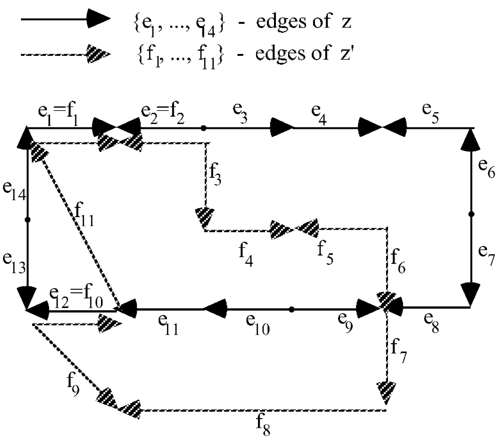

Figure 1.

(c-) $\Rightarrow$ (a-) and (c) $\Rightarrow$ (a): Assume that (c-) holds. As $z^2 = z - z^1$ is a cycle and coordinates of cycles are restricted to $-1, 0$ and $+1$, we have that $\text{supp}(z^2) = [\text{supp}(z)\backslash\text{supp}(z^1)] \cup [\text{supp}(z^1)\backslash\text{supp}(z)]$. It follows that $\text{supp}(z) \neq \text{supp}(z^1)$ (for otherwise $\text{supp}(z^2) = \emptyset$), and that $\text{supp}(z) \cap \text{supp}(z^1) \neq \emptyset$ (for otherwise the support of z^2 would equal the union of the supports of two cycles, namely, z and z^1). Finally, as $\text{supp}(z^1)\backslash\text{supp}(z) \subseteq \text{supp}(z^2)$, we have that $\text{supp}(z^1)\backslash\text{supp}(z) \subseteq \text{supp}(z^1) \cap \text{supp}(z^2) \subseteq U'$, the latter by condition (ii) of (c). We have shown that $z' \equiv z^1$ satisfies conditions (i), (ii) and (iv) of (a), establishing (a-). Finally, trivially, if z^1 satisfies condition (iii) of (c) then z' satisfies condition (iii) of (a). So, we also have (c) $\Rightarrow$ (a).

(b) $\Rightarrow$ (c): Assume that (b) holds. The sequences $h_1, h_2, \ldots, h_{s-1}, h_s, e_{p+1}, \ldots, e_q, e_1, \ldots, e_m$, and $h_s, h_{s-1}, \ldots, h_2, h_1, e_{m+1}, \ldots, e_p$, are then the supports of two cycles, say z^1 and z^2, respectively, with $z^1_{e_i} = z_{e_i}$ for $i \in \{p+1, \ldots, q, 1, \ldots, m\}$, with $z^2_{e_i} = z_{e_i}$ for $i \in \{m+1, m+2, \ldots, p\}$ and with $z^1_{h_i} = -z^2_{h_i}$ for $i\{1, \ldots, s\}$. In particular, we have $z = z^1 + z^2$ and $z^t_e z_e \geqslant 0$ for $t = 1, 2$ and every $e \in U$, that is, conditions (i) and (iii) of (c) are satisfied. Further, for $e \in U$ with $z^1_e \neq 0$ and $z^2_e \neq 0$, we have that $e \in \{h_1, h_2, \ldots, h_s\}$, implying that $z_e = 0$ and $e \in U'$. So, condition (ii) of (c) is also satisfied. □

Lemma 7 assures that, given a cycle z, the existence of a cycle z' satisfying (a-) implies the existence of a cycle z' that satisfies (a), and the existence of cycles z^1 and z^2 that satisfy (c-) implies the existence of cycles z^1 and z^2 that satisfy (c). It is noted, however, that a cycle z' that satisfies (a-) need not satisfy (a), and cycles z^1 and z^2 that satisfy (c-) need not satisfy (c) (and no such claims are made in Lemma 7).

Lemma 7 provides alternatives to the "nonexistence of z'"-condition in Theorem 5. Consequently, we get the following two characterizations of directions of 1-dimensional faces of network polyhedra.

THEOREM 8. *Suppose x is a vertex of the network polyhedron P. Then $y \in P\backslash\{x\}$ lies in a 1-dimensional face of P that contains x if and only if y has a representation $y = x + \alpha z$ with $\alpha > 0$ and with z a cycle which cannot be expressed as $z = z^1 + z^2$ where $supp(z^1) \cap supp(z^2) \subseteq float(x)\backslash supp(z)$.*

Given a cycle z and a set of arcs U' we say that *U' can be used to bisect z* if condition (b) of Lemma 7 is satisfied.

THEOREM 9. *Suppose x is a vertex of the network polyhedron P. Then $y \in P\backslash\{x\}$ lies in a 1-dimensional face of P that contains x if and only if y has a representation $y = x + \alpha z$ with $\alpha > 0$ where z is a cycle and float(x) cannot be used to bisect z.*

Proposition 3 implies that every direction of a 1-dimensional face of a network polyhedron P that contains a given vertex x, is a scalar multiple of a cycle z for which there is a positive scalar α, such that $x+\alpha z \in P$. We observe that the latter is equivalent to the assertion that

$$[(z_e > 0) \Rightarrow (x_e < U_e)] \quad \text{and} \quad [(z_e < 0) \Rightarrow (x_e > L_e)] \quad \text{for each} \quad e \in A \tag{12}$$

(see the second comment following Theorem 5). Cycles z that satisfy (12) can be determined from the *nonnegative* cycles of the network with node-set N, with arc-set $A'' \cup A'''$ where $A'' \equiv \{e \in A : x_e < U_e\}$ and $A''' \equiv \{e = (j,i) \in A : x_{ij} > L_{ij}\}$ and with all lower bounds 0 and all upper bounds 1; specifically, a nonnegative cycle in this network defines a cycle satisfying (12) by reversing the sign of the arcs in A''' (it is noted that $A'' \cup A'''$ has duplicate arcs when $A'' \cap A''' \neq \emptyset$). Theorem 8 further shows that a cycle satisfying (12) is a direction of a 1-dimensional face containing x if and only if there is no decomposition of z as a sum of two cycles z^1 and z^2, where $\mathrm{supp}(z^1) \cap \mathrm{supp}(z^2) \subseteq \mathrm{float}(x) \backslash \mathrm{supp}(z)$ (and another characterization follows from Theorem 9).

We next examine cycles that correspond to directions of 1-dimensional faces in a particular example.

EXAMPLE 1: CONSTRAINED-SHAPE PARTITIONING PROBLEMS

Consider the (transportation) network whose graph is demonstrated in Figure 2 below, with the lower and upper bounds $l_{(r,n+s)} = 0$ and $u_{(r,n+s)} = 1$ for $1 \leqslant r \leqslant n$ and $1 \leqslant s \leqslant p$ and $l_{(n+s,n+p+1)}$ and $u_{(n+s,n+p+1)}$ prescribed arbitrarily for $1 \leqslant s \leqslant p$ and with right-hand side vector b with $b_r = -1$ for $r = 1, \ldots, n$, $b_{s+n} = 0$ for $s = 1, \ldots, p$ and with $b_{n+p+1} = n$. For brevity, we let for $s = 1, \ldots, p$, $l_s \equiv l_{(n+s,n+p+1)}$ and $u_s \equiv u_{(n+s,n+p+1)}$.

With b integral, a vertex of the corresponding network polyhedron is known to be integral (e.g., [9]). In particular, for each vertex x, we have that for every $r = 1, \ldots, n$, $x_{(r,n+s)} = 1$ for exactly $s \in \{1, \ldots, p\}$, and for each

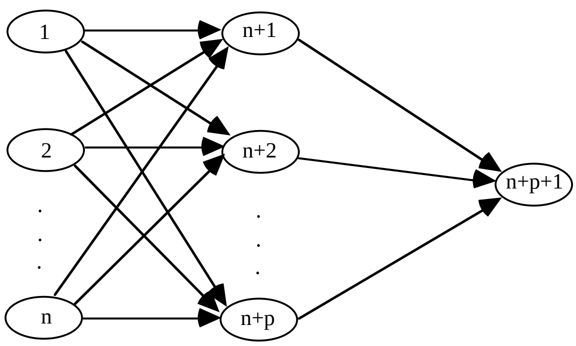

Figure 2.

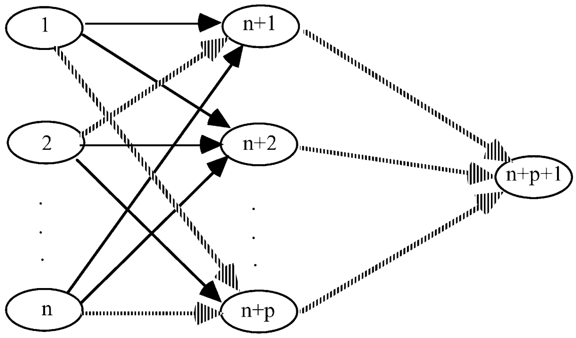

Figure 3.

$s = 1, \ldots, p$, $x_{(n+s,n+p+1)}$ is the number of nodes $r \in \{1, \ldots, n\}$ with $x_{(r,n+s)} = 1$. Thus, a vertex corresponds to an assignment of nodes $1, \ldots, n$ to the p destinations $n+1, n+2, \ldots, n+p$, subject to requirements/capacity constraints on the number of nodes assigned to each destination; so, vertices correspond to *partitions* of $\{1, \ldots, n\}$ into parts, indexed by $1, \ldots, p$ subject to lower and upper bounds. In Figure 3 above, we illustrate the support of a vertex using the network representation of Figure 2. It is observed that for a vertex $x, n+p+1$ occurs in every arc in float(x).

We shall use the standard representation of cycles through sequences of nodes. Given a vertex x, a cycle z for which $x + \alpha z$ is feasible (that is, in the network polyhedron) has either of the following two representations:

(i) **A cycle that excludes $n+p+1$**: For some $2 \leqslant k \leqslant p$, there are sequences $r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ of distinct elements from $\{1, \ldots, n\}$ and from $\{1, \ldots, p\}$, respectively, and z is represented by the sequence $r_1, n+s_1, r_2, n+s_2, \ldots, r_k, n+s_k$; this representation corresponds to a cyclic change where for $j = 1, \ldots, k, r_j$ is moved from part s_{j-1} to part s_j. The requirement that $x + \alpha z \in P$ for some $\alpha > 0$ imposes the constraint $x_{r_j, n+s_{j-1}} = 1$ for $j = 1, \ldots, k$ (with $s_0 = s_k$); thus, $s_1, \ldots, s_k$ are determined by $r_1, \ldots, r_k$ and the latter sequence characterizes z.

(ii) **A cycle that includes $n+p+1$**: For some $1 \leqslant k \leqslant p$, there are sequences $r_2, \ldots, r_k$ and $s_1, \ldots, s_k$ of distinct elements from $\{1, \ldots, n\}$ and from $\{1, \ldots, p\}$, respectively, and z is represented by the sequence $n+p+1, n+s_1, r_2, n+s_2, \ldots, r_k, n+s_k$; this representation corresponds to a sequential change where for $j = 2, \ldots, k, r_j$ is moved from part s_{j-1} to part s_j, with the assignment of part s_1 lowered by one element and the assignment of part s_k increased by one element. The requirement that $x + \alpha z \in P$ for some $\alpha > 0$ imposes the constraints $x_{r_j, n+s_{j-1}} = 1$ for $j = 2, \ldots, k$, $x_{n+s_1, n+p+1} > ls_1$ and $x_{n+s_k, n+p+1} < u_{s_k}$; thus, $s_1, \ldots, s_{k-1}$ are determined by $r_2, \ldots, r_k$ and $r_2, \ldots, r_k, s_k$ characterizes z.

With p fixed, the number of cycles z with the first representation is then bounded by $\sum_{k=1}^{p} \binom{n}{k}(k-1)! = O(n^p)$ (we accounted for permutation-invariance of cycle-representation), and the number of cycles z with the second representation is bounded by $\sum_{k=1}^{p-1} \binom{n}{k} p = O(n^p)$. The total number of cycles is then $O(n^p)$.

Theorem 9 shows that a cycle z is a direction of a 1-dimensional face that contains vertex x if and only if float$(x)\setminus$supp(z) cannot be used to bisect z. This requirement provides a test for a cycle to be a direction of a 1-dimensional face that contains vertex X; further, the requirement can be used to tighten the bound on such directions. As $n+p+1$ occurs in every arc in float(z), condition (b) of Lemma 7 means that:

Under (i): There exist no pairs of elements s_u and s_v in $\{1, \ldots, k\}$ with $l_{su} < x_{n+s_u,n+p+1} < u_{s_u}$ and $l_{s_v} < x_{n+s_v,n+p+1} < u_{s_v}$.

Under (ii): There exist no element s_u in $\{1, \ldots, k\}$ with $l_{s_u} < x_{n+s_u,n+p+1} < u_{s_u}$. For arbitrary l_s's and u_s's, we have no general expression to account for the cycles satisfying the above requirement, and the bound we get on the number of cycles is $O(n^p)$.

We next consider bounds on the number of edge-directions without reference to a particular vertex x that occurs in the 1-dimensional face. In cases where no prior information is available for edges which are necessarily in float(x) for vertices x, the bound on the number of $\sim$-edge-directions is the number of cycles. The total number of cycles under either (i) or (ii) is $\sum_{k=2}^{p} \binom{n}{k}\binom{p}{k} k!(k-1)!$. So, the bound on the number of $\sim$-edge-directions is $2\sum_{k=2}^{p} \binom{n}{k}\binom{p}{k} k!(k-1)!$ □

EXAMPLE 2: OPEN-SHAPE PARTITIONING PROBLEMS Consider Example 1 with $u_s = n+1$ and $l_s = 0$ for every s, that is, there are no upper bounds on part-capacities. In this case, every arc $(n+s, n+p+1)$ is in float(x) and the only case where float(x) cannot be used to bisect z is for cycles of the second form with $k=2$. These cycles correspond to a single switch of one element from one part to another. The number of such cycles is then $n\binom{p}{2} = O(n)$. Consequently, the number of $\sim$-edge-directions for the corresponding partition polytopes is bounded by $n\binom{p}{2}$. □

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